

**BARNHART**

**Theory of**

**Correlation among Constituents**

**Mathematics**

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# THEORY OF CORRELATION AMONG CONSTITUENTS

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BY

CHARLES ANTHONY BARNHART  
B. A. UNIVERSITY OF ILLINOIS, 1905

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## THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF  
MASTER OF ARTS  
IN MATHEMATICS

IN

THE GRADUATE SCHOOL  
OF THE  
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THE GRADUATE SCHOOL

June 1 , 1911

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Charles Anthony Barnhart

ENTITLED - "Theory of Correlation among Constituents."

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of Arts in Mathematics.

*H. L. Riets*

In Charge of Major Work

*E. Townsend*

Head of Department

Recommendation concurred in:

\_\_\_\_\_

\_\_\_\_\_

} Committee

on

} Final Examination



## Contents.

Article.	Page.
1. Introduction.....	1.
2. Assumption of Random Sampling.....	2.
3. Special Cases.....	3.
4. Numerical Cases.....	5.
5. Mathematical Theory.....	6.
6. Special Cases.....	8.
7. Signs of Correlation Co-efficients.....	10.
8. Summary.....	12.
Correlation Tables.....	13-22.



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## Theory of Correlation among Constituents .

1. Introduction. Consider a material body  $A$ , which is composed of a number of constituents. To be concrete, a body may contain chemical constituents such as protein, oil, ash and carbo-hydrates in certain quantities. It is known, in some cases, that such constituents fluctuate in value from one individual to another of the same class. Let  $A_1, A_2, \text{-----}, A_n$  be a class of such individuals. The question naturally arises, upon selecting at random another body  $A_2$  from this class in which one constituent is relatively high, as to whether the other constituents shall be relatively high or relatively low. The discussion of this question of associated fluctuations of constituents leads to what we have termed the theory of correlation among constituents.

A constituent may be considered from either of two viewpoints. In the first place, merely the absolute values of the constituents need be considered, and the resulting correlation co-efficients employed as the basis of an investigation. On the other hand, the constituents may be viewed as proportional parts of the entire complex, in which case the sum of the constituents is unity.

The latter viewpoint is chosen as the one upon which to base the present investigation, since it permits the constituents to be viewed as frequency groups whose sum is constant and equal to unity. Prof. Karl Pearson has made a study of the correlations existing between frequency groups whose sum is constant and his article furnished valuable suggestions by means of which to begin



the investigation of the problem which we have set.\*

2. Assumption of random sampling. Suppose that a substance is composed of  $n$  constituents,  $y_1, y_2, \dots, y_n$ . Then

$$y_1 + y_2 + \dots + y_n = 1.$$

Moreover, let it be assumed that the law of random sampling is in operation. Then, if an increase is assigned to one constituent, the most probable manner in which the resulting decrease, ( which must occur among the remaining constituents taken together as a whole ), will be distributed among these constituents is in proportion to their respective values.

If the correlation between two constituents,  $y_r$  and  $y_s$ , is desired, we may assign a variation to  $y_r$ . As a result,  $y_s$  will suffer a variation opposite in sign to that of  $y_r$ , and such that

$$\delta y_s = -\delta y_r \frac{y_s}{1-y_r}$$

Then

$$\delta y_r \delta y_s = -\delta^2 y_r \frac{y_s}{1-y_r}$$

Summing for all samplings,

$$n \sigma_r \sigma_s r_{y_r y_s} = -n \sigma_r^2 \frac{y_s}{1-y_r}$$

or

$$r_{y_r y_s} = -\frac{\sigma_r}{\sigma_s} \frac{y_s}{1-y_r} \quad (1)$$

Again, we may write

$$\delta y_r = -\delta y_s \frac{y_r}{1-y_s}$$

---

\* "On the Probable Error of Frequency Constants."



Then

$$\delta y_s \delta y_r = - \delta^2 y_s \frac{y_r}{1-y_s}$$

Summing for all samplings, and dividing by the number of them,

$$r_{y_s, y_r} = - \frac{G_s}{G_r} \frac{y_r}{1-y_s} \quad (2)$$

Since in obtaining the relations (1) and (2), the total variation caused in either of these two constituents by a change in the other has been considered, the two correlation co-efficients are equal.

Therefore,

$$\frac{G_r}{G_s} \cdot \frac{y_s}{1-y_r} = \frac{G_s}{G_r} \cdot \frac{y_r}{1-y_s}$$

from which it is evident that

$$\frac{G_r}{G_s} = \frac{\sqrt{y_r(1-y_r)}}{\sqrt{y_s(1-y_s)}} \quad (3)$$

or

$$G_r = K \sqrt{y_r(1-y_r)} \quad \text{and} \quad G_s = K \sqrt{y_s(1-y_s)} \quad (4)$$

Substituting these values in (1) and (2), respectively,

$$r_{y_r, y_s} = r_{y_s, y_r} = - \sqrt{\frac{y_r y_s}{(1-y_r)(1-y_s)}} \quad (5)$$

from which it appears that all the correlation co-efficients are negative.

3. Special cases. If there are only two constituents in a substance, then

$$r_{y_1, y_2} = - \sqrt{\frac{y_1 y_2}{(1-y_1)(1-y_2)}} = - \sqrt{\frac{y_1 y_2}{y_2 y_1}} = -1. \quad (6)$$

That is, there is a perfect negative correlation, no matter in what proportions the two constituents enter into the complex. If a



larger number of constituents is considered such a relation no longer exists. There are, however, several interesting special cases that should be mentioned.

It is evident from relations (4) and (5) that if the constituents are practically equal that the standard deviations are equal, and that all of the correlation co-efficients are, like-wise, equal. That is, if each of the constituents fluctuates very closely about the mean value  $\frac{1}{n}$ , then

$$r_{Y_i, Y_j} = -\sqrt{\frac{\frac{1}{n} \cdot \frac{1}{n}}{\left(\frac{n-1}{n}\right)\left(\frac{n-1}{n}\right)}} = -\frac{1}{n-1} \quad (7)$$

This reduces, for the case of two constituents, to  $-1$ ; for three equal constituents to  $-\frac{1}{2}$ ; and so on.

Moreover, if the  $n$  constituents are so related that the product of two of them divided by the sum of the remaining constituents is constant, then the correlation between the first two remains constant. For example, if

$$\frac{Y_1 Y_2}{1 - Y_1 - Y_2} = b$$

then

$$\begin{aligned} r_{Y_1, Y_2} &= -\sqrt{\frac{Y_1 Y_2}{(1 - Y_1)(1 - Y_2)}} = -\sqrt{\frac{Y_1 Y_2}{1 - Y_1 - Y_2 + Y_1 Y_2}} \\ &= -\sqrt{\frac{1}{\frac{1 - Y_1 - Y_2}{Y_1 Y_2} + 1}} = -\sqrt{\frac{1}{\frac{1}{b} + 1}} \\ &= -\sqrt{\frac{b}{1 + b}} \quad (8) \end{aligned}$$

Again, if the sum of  $n-2$  constituents is constant, then the maximum correlation between the remaining two results for equal values of



these two constituents, and a minimum when either of them is practically equal to zero. For, if the sum of the n-2 constituents be represented by a, then

$$r_{y_1, y_2} = -\sqrt{\frac{y_1 y_2}{(1-y_1)(1-y_2)}} = -\sqrt{\frac{y_1 y_2}{1-y_1-y_2+y_1 y_2}}$$

$$= -\sqrt{\frac{y_1 y_2}{a+y_1 y_2}} = -\sqrt{1-\frac{a}{a+y_1 y_2}} \quad (9)$$

The expression under the radical sign is a maximum for  $y_1 = y_2$ , and a minimum for each practically equal to zero.

4. Numerical cases. At this point correlation tables were constructed and the correlation co-efficients computed for the case of a substance composed of five constituents.\* These constituents are represented in the tables by the numbers 1,2,3,4, and 5. (See Pp. 13- 22 ). Examination of these tables show that the correlation co-efficients are not all negative as is the case under the assumption of random sampling. Arranging the co-efficients in two columns, and placing the values of the correlation co-efficients obtained from the tables in the first column and those obtained from the formula

$$r_{y_r, y_s} = -\sqrt{\frac{y_r y_s}{(1-y_r)(1-y_s)}}$$

in the second; a large discrepancy appears.

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\* The data employed in the tables was furnished through the courtesy of Dr. H. S. Grindley, and are the results of a series of experiments conducted by the Laboratory of Physiological Chemistry of the Department of Animal Husbandry which are, as yet, unpublished.



$r_{12}$	-0.125	-0.467
$r_{13}$	-0.308	-0.574
$r_{14}$	-0.135	-0.095
$r_{15}$	-0.815	-0.488
$r_{23}$	+0.144	-0.533
$r_{24}$	+0.164	-0.028
$r_{25}$	-0.143	-0.045
$r_{34}$	+0.333	-0.034
$r_{35}$	-0.008	-0.056
$r_{45}$	-0.091	-0.029

This discrepancy may be explained by calling attention to the fact that when the absolute weights of the constituents, which had been chosen at random, were divided by the absolute weights of their respective samples that a set of values was obtained which did not exhibit random sampling. Prof. Karl Pearson has attributed this discrepancy, which occurs when indices are formed from absolute values, to what he terms "spurious correlation". It is evident, therefore, that theory of correlation among constituents must be developed on some assumption other than that of random sampling. It may be developed upon a purely mathematical basis from the mathematical definition of a correlation co-efficient and the assumption that the sum of the constituents is always unity.

5. Mathematical theory. If the deviations of any two constituents from their mean values be represented by  $Y_1$  and  $Y_2$ , respectively, then a mathematical definition of a correlation co-efficient is given by

$$r_{12} = \frac{\sum Y_1 Y_2}{n \sigma_1 \sigma_2}$$

where  $\sum Y_1^2 = n \sigma_1^2$  and  $\sum Y_2^2 = n \sigma_2^2$ .



Suppose that a substance is composed of  $n$  constituents  $y_1, y_2, \dots, y_k, \dots, y_s, \dots, y_n$ , where

$$y_1 + y_2 + \dots + y_k + \dots + y_s + \dots + y_n = 1.$$

Now, if the means of these constituents be represented by  $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_k, \dots, \bar{y}_s, \dots, \bar{y}_n$ , respectively, we may transform co-ordinates to a new system which has its origin at the means of the  $n$  constituents. If the deviations from these means be denoted by  $Y_1, Y_2, \dots, Y_k, \dots, Y_s, \dots, Y_n$ , respectively, then

$$Y_1 + y_1 + Y_2 + y_2 + \dots + Y_k + y_k + \dots + Y_s + y_s + \dots + Y_n + y_n = 1.$$

But

$$\bar{y}_1 + \bar{y}_2 + \dots + \bar{y}_k + \dots + \bar{y}_s + \dots + \bar{y}_n = 1$$

since it is sum of the mean values of values whose sum is constantly equal to 1. Now, by definition

$$\begin{aligned} r_{ks} &= \frac{\sum Y_k Y_s}{n \sigma_k \sigma_s} = \frac{\sum Y_k (-Y_1 - Y_2 - \dots - Y_{k-1} - Y_{k+1} - \dots - Y_{s-1} - Y_{s+1} - \dots - Y_n)}{n \sigma_k \sigma_s} \\ &= - \left( \frac{\sum Y_k Y_1 + \sum Y_k Y_2 + \dots + \sum Y_k Y_{k-1} + \sum Y_k^2 + \sum Y_k Y_{k+1} + \dots + \sum Y_k Y_{s-1} + \sum Y_k Y_{s+1} + \dots + \sum Y_k Y_n}{n \sigma_k \sigma_s} \right) \\ &= -n \left( \frac{\sigma_1 \sigma_k r_{k,1} + \sigma_2 \sigma_k r_{k,2} + \dots + \sigma_{k-1} \sigma_k r_{k,k-1} + \sigma_k^2 + \sigma_k \sigma_{k+1} r_{k,k+1} + \dots + \sigma_k \sigma_{s-1} r_{k,s-1} + \sigma_k \sigma_{s+1} r_{k,s+1} + \dots + \sigma_k \sigma_n r_{k,n}}{n \sigma_k \sigma_s} \right) \\ &= - \left( \frac{\sigma_1 r_{k,1} + \sigma_2 r_{k,2} + \dots + \sigma_{k-1} r_{k,k-1} + \sigma_k + \sigma_{k+1} r_{k,k+1} + \dots + \sigma_{s-1} r_{k,s-1} + \sigma_{s+1} r_{k,s+1} + \dots + \sigma_n r_{k,n}}{\sigma_s} \right) \end{aligned}$$

from which

$$\sigma_1 r_{k,1} + \sigma_2 r_{k,2} + \dots + \sigma_n r_{k,n} = \sigma_k.$$



Likewise, if the value of  $Y_k$  be substituted in  $\sum \frac{Y_k Y_s}{n \sigma_k \sigma_s}$  the relation becomes

$$\sigma_1 r_{s,1} + \sigma_2 r_{s,2} + \dots + \sigma_n r_{s,n} = -\sigma_s$$

That is, in the case of  $n$  constituents, the  $\frac{n(n-1)}{2}$  correlation coefficients satisfy the  $n$  relations

$$\left. \begin{aligned} \sigma_2 r_{12} + \sigma_3 r_{13} + \dots + \sigma_n r_{1,n} &= -\sigma_1 \\ \sigma_1 r_{21} + \sigma_3 r_{23} + \dots + \sigma_n r_{2,n} &= -\sigma_2 \\ \dots &\dots \\ \sigma_1 r_{n,1} + \sigma_2 r_{n,2} + \dots + \sigma_{n-1} r_{n,n-1} &= -\sigma_n \end{aligned} \right\} \quad (10)$$

6. Special cases. In the case of two constituents these relations take the form

$$\sigma_1 r_{21} = -\sigma_2$$

$$\sigma_2 r_{12} = -\sigma_1$$

or

$$r_{21} = -\frac{\sigma_2}{\sigma_1} \quad \text{and} \quad r_{12} = -\frac{\sigma_1}{\sigma_2}$$

from which

$$\frac{\sigma_2}{\sigma_1} = \frac{\sigma_1}{\sigma_2}$$

or

$$\sigma_2 = \sigma_1$$

and

$$r_{21} = r_{12} = -1 \quad (11)$$

which agrees with the result obtained under the assumption of random sampling.

In the case of three constituents

$$\left. \begin{aligned} \sigma_2 r_{12} + \sigma_3 r_{13} &= -\sigma_1 \\ \sigma_1 r_{21} + \sigma_3 r_{23} &= -\sigma_2 \\ \sigma_1 r_{31} + \sigma_2 r_{32} &= -\sigma_3 \end{aligned} \right\}$$



from which

$$\left. \begin{aligned} r_{12} &= \frac{\sigma_3^2 - \sigma_1^2 - \sigma_2^2}{2 \sigma_1 \sigma_2} \\ r_{13} &= \frac{\sigma_2^2 - \sigma_1^2 - \sigma_3^2}{2 \sigma_1 \sigma_3} \\ r_{23} &= \frac{\sigma_1^2 - \sigma_2^2 - \sigma_3^2}{2 \sigma_2 \sigma_3} \end{aligned} \right\} \quad (12)$$

In the case of four or more constituents, it is impossible to solve the n independent equations for the  $\frac{n(n-1)}{2}$  correlation co-efficients in terms of the standard deviations alone. That is, the values of each and every correlation co-efficient depends upon the values of certain other correlation co-efficients. For instance, for  $n=4$ , there are six correlation co-efficients and only four relations by means of which to determine them; for  $n = 5$ , there are only five relations by means of which to determine ten correlation co-efficients.

Returning to (12) it is rather interesting to note that if the values  $\sigma_1 = k \sqrt{y_1(1-y_1)}$  ;  $\sigma_2 = k \sqrt{y_2(1-y_2)}$  and

$$\sigma_3 = k \sqrt{y_3(1-y_3)}$$

which were obtained under the assumption of random sampling, be substituted in relations (12), then

$$\begin{aligned} r_{12} &= - \sqrt{\frac{y_1 y_2}{(1-y_1)(1-y_2)}} \\ r_{13} &= - \sqrt{\frac{y_1 y_3}{(1-y_1)(1-y_3)}} \\ r_{23} &= - \sqrt{\frac{y_2 y_3}{(1-y_2)(1-y_3)}} \end{aligned}$$



which values are identical with those obtained under the assumption of random sampling.

Moreover, if in (10) we assume all the standard deviations to be equal, all the values of the correlation co-efficients are equal to  $\frac{-1}{n-1}$ , which is the value obtained in Art. 3 under the assumption of random sampling for equal values of the constituents and the resulting equal values of the standard deviations.

7. Signs of the correlation co-efficients. From (10), since the standard deviations are always taken positively, the apparent conclusion is that all of the correlation co-efficients may be negative. This may be proven in the following manner;

Let the  $n$  constituents of a substance be represented by  $y_1, y_2, \dots, y_n$ . Suppose that these constituents in a certain individual take the definite values  $a_1, a_2, \dots, a_n$ , respectively, and that these values have been permuted. In other words, we have  $n$  samples, such that their constituents satisfy the above conditions. Each value is, therefore, assigned to the  $y_i$ -constituent  $n-1$  times so that the mean of the  $y_i$ -constituent is  $\frac{n-1(a_1 + a_2 + \dots + a_n)}{n}$ .

But

$$a_1 + a_2 + \dots + a_n = 1$$

so that the mean of the ~~of the~~  $y_i$ -constituent is, therefore,  $\frac{1}{n}$ .

The mean of each constituent is, therefore,  $\frac{1}{n}$ , since the constituents are identical in sum.

The correlation between any two constituents, which is given by  $r_{k,s} = \frac{\sum (y_k - \frac{1}{n})(y_s - \frac{1}{n})}{n \sigma_k \sigma_s}$ , will have the same sign as  $\sum (y_k - \frac{1}{n})(y_s - \frac{1}{n})$

since all values involved are absolute values. On account of the



fact that the distribution of each constituent has been built up by permuting the  $\underline{n}$  constituents  $\underline{n}$  at a time, the sum of the product deviations in each of the  $\frac{n(n-1)}{2}$  different cases is the same.

Therefore, if any one of the correlation co-efficients is shown to be negative, then all of them are equal and negative in sign.

Let us consider the  $y_1$  - and  $y_2$  -constituents. Let  $a_1$  remain fixed as the  $y_1$  -constituent, then a permutation of the  $\underline{n-1}$  remaining values will cause each of them to be the  $y_2$  -constituent  $\underline{n-2}$  times. Then for all samples for which  $a_1$  is the  $y_1$  -constituent the product deviations will add up to

$$\begin{aligned} & \underline{n-2} \left( a_1 - \frac{1}{n} \right) \left( a_2 - \frac{1}{n} - a_3 - \frac{1}{n} - \dots - a_n - \frac{1}{n} \right) \\ &= \underline{n-2} \left( a_1 - \frac{1}{n} \right) \left( 1 - a_1 - \frac{n-1}{n} \right) = \underline{n-2} \left( a_1 - \frac{1}{n} \right) \left( \frac{1}{n} - a_1 \right) = - \underline{n-2} \left( a_1 - \frac{1}{n} \right)^2 \end{aligned}$$

Likewise, the sum of the product deviations for all cases for which  $a_2$  is the  $y_1$  -constituent is given by  $- \underline{n-2} \left( a_2 - \frac{1}{n} \right)^2$ .

Therefore,

$$\sum \left( y_1 - \frac{1}{n} \right) \left( y_2 - \frac{1}{n} \right) = - \underline{n-2} \left\{ \left( a_1 - \frac{1}{n} \right)^2 + \left( a_2 - \frac{1}{n} \right)^2 + \dots + \left( a_n - \frac{1}{n} \right)^2 \right\}$$

is negative in sign and, therefore, all of the correlation co-efficients are negative.

Again, from (10) it is evident that all of the correlations co-efficients can not be positive. Moreover, it is evident that the correlations of each constituent with each of the remaining constituents can not all be positive. If the various groups of correlation co-efficients be written down which contain the correlations of each constituent with all the remaining constituents, there results  $\underline{n}$  such groups, each containing  $\underline{n-1}$  correlation co-efficients. At least, one correlation co-efficient in each group must be negative



and since each correlation co-efficient appears in two, and only two, groups there can not be less than  $\frac{n}{2}$  negative correlation co-efficients if  $n$  is even, and not less than  $\frac{n+1}{2}$  negative correlation co-efficients if  $n$  is odd.

8. Summary. (a) Under the assumption of random sampling a theory is developed from which all the correlation co-efficients connecting constituents are negative.

(b) Computation of the correlation co-efficients from tables by the usual method give values which do not arise under the assumption of random sampling.

(c) A mathematical theory is developed from the mathematical definition of a correlation co-efficient which is consistent with computed values of the correlation co-efficients. For the case of two constituents or for equal standard deviations, the results agree with those obtained from the assumption of random sampling as made in Art. 2.

(d) All of the correlation co-efficients may be negative.

(e) All of the correlation co-efficients can not be positive.

(f) If  $n$  is even, not less than  $\frac{n}{2}$  of the correlation co-efficients can be negative; if  $n$  is odd not less than  $\frac{n+1}{2}$  of the correlation co-efficients can be negative.



No. 1.

No. 2.

	.77875	.79125	.80375	.81625	.82875	.84125	.85375	.86625	.87875	.89125	.90375	
.024							1				1	2
.028					1		1					2
.032		1	1	1	3	3	3		1.5			13.5
.036			8.5	5	4	6	3	6.5	3	1.5		37.5
.040	3	4	5.5	10	11	8	4	7.5	3.5	0.5		57
.044	2	3	5	8	15.5	11.5	9	5.5	3	2		64.5
.048		1	7	3	6	5	4	0.5	1			27.5
.052		2	1	2	1	1	2	2				11
.056						2						2
.060												0
.064					1							1
	5	11	28	29	42.5	36.5	27	22	12	4	1	218

$$M_1 = .83405. \quad M_2 = .04161.$$

$$\sigma_1 = .0261. \quad \sigma_2 = .00569.$$

$$r_{12} = -0.1252.$$



No. 1.

No. 3.

	.77875	.79125	.80375	.81625	.82875	.84125	.85375	.86625	.87875	.89125	.90375	
.0465					1	1			0.5			2.5
.0495						1	1	2.5	0.5			5
.0525	1		1	3	1.5	5.5	5	1.5	3	2		23.5
.0555			2	7	7.5	2.5	2	3	3	2	1	30
.0585	1	2	8	2	7	3	1	4.5	2.5			31
.0615		1	5	4	5	11	1	3.5	2.5			33
.0645		1	4	6	7	3	10	6				37
.0675		4	5	4	6	6	5	1				31
.0705	1	1	1	2	5.5	1.5						12
.0735	1	1	2			1	2					7
.0765	1	1		1	1	1						5
.0795					1							1
	5	11	28	29	42.5	36.5	27	22	12	4	1	218

$$M_1 = .83405. \quad M_3 = .0615.$$

$$\sigma_1 = .0261. \quad \sigma_3 = .0066.$$

$$r_{13} = -0.3077.$$



No. 1.

No. 4.

	.77875	.79125	.80375	.81625	.82875	.84125	.85375	.86625	.87875	.89125	.90375	
.012	1				1		1	1				4
.014		1	1	3.5	3	1	2	3	2			16.5
.016		2.5	9.5	11	14	8.5	7	8	6	3	1	70.5
.018	2	4.5	10.5	9.5	20	17	13	8	2	1		87.5
.020	1.5	1	4	2	3	6.5	3	1	1			23
.022	0.5	1	2.5	3		2		1	1			11
.024		1	0.5		0.5	1.5	1					4.5
.026												0
.028					1							1
	5	11	28	29	42.5	36.5	27	22	12	4	1	218

$$M_1 = .83405. \quad M_4 = .01752.$$

$$\sigma_1 = .0261. \quad \sigma_4 = .00232.$$

$$r_{14} = -0.1354.$$



No. 1.

No. 5.

	.77875	.79125	.80375	.81625	.82875	.84125	.85375	.86625	.87875	.89125	.90375	
.005							4	3	4	3	1	15
.015	1					3	4	10	5			23
.025				.1	3.5	7.5	10	8	1			31
.035				1	3	12	4	1	1			22
.045			1	2	16	11	2.5		1	1		34.5
.055			2	9	14	2	0.5					27.5
.065		3	10	8	3	1	1					26
.075		3	8.5	7	3							21.5
.085	2	4	6.5	1								13.5
.095	1						1					2
.105	1	1										2
	5	11	28	29	42.5	36.5	27	22	12	4	1	218

$$M_1 = .83405. \quad M_5 = .04532.$$

$$\sigma_1 = .0261. \quad \sigma_5 = .024.$$

$$r_{15} = -0.8153.$$



No. 2.

No.	.024	.028	.032	.036	.040	.044	.048	.052	.056	.060	.064	
.0465			0.25	1.25		1						2.5
.0495		1	0.25	0.25	1	1.5		1				5
.0525			3	4.5	7	6.5	1.5		1			23.5
.0555	1		3	6	3.5	12.5	3	1				30
.0585			3	7.25	8.75	7	4				1	31
.0615		1	2	4.25	9.75	10	3	2	1			33
.0645	1		1	5.5	6.5	15	4	4				37
.0675			1	5.5	10.5	5	7	2				31
.0705				1	7	2	1	1				12
.0735				2	2	2	1					7
.0765						2	3					5
.0795					1							1
	2	2	13.5	37.5	57	64.5	27.5	11	2	0	1	218

$$M_2 = .04161. \quad M_3 = .0615.$$

$$\sigma_2 = .00569. \quad \sigma_3 = .0066.$$

$$r_{23} = 0.1438.$$



No. 2.

No. 4.

	.024	.028	.032	.036	.040	.044	.048	.052	.056	.060	.064	
.012			2		1	1						4
.014			4.5	2.5	3	4.5	2					16.5
.016	1	1	3.5	15	20	22.5	5.5	2				70.5
.018	1	1	2.5	13.5	21	24.5	14	8	2			87.5
.020			1	3.5	4.5	9	3	1			1	23
.022				3	4	2	2					11
.024					2.5	1	1					4.5
.026												0
.028					1							1
	2	2	13.5	37.5	57	64.5	27.5	11	2	0	1	218

$$M_2 = .04161. \quad M_4 = .01752.$$

$$\sigma_2 = .00569. \quad \sigma_4 = .00232.$$

$$r_{24} = 0.1644.$$



No. 2.

No. 5.

	.024	.028	.032	.036	.040	.044	.048	.052	.056	.060	.064	
.005	2			1	1	6	2	3				15
.015			1	5	8	6	2	1				23
.025			0.5	7	6.5	7.5	7.5		1		1	31
.035				2	5	10	2	2	1			22
.045		0.5	5	4	8	10	5	2				34.5
.055		0.5		6	11	9		1				27.5
.065		1	1	5	5	5	8	1				26
.075			4	6	3.5	6	1	1				21.5
.085			2	1.5	7	3						13.5
.095					1	1						2
.105					1	1						2
	2	2	13.5	37.5	57	64.5	27.5	11	2	0	1	218

$$M_2 = .04161. \quad M_5 = .04532.$$

$$\sigma_2 = .00569. \quad \sigma_5 = .024.$$

$$r_{25} = -0.1429.$$



No. 3.

No. 4.													
	.0465	.0495	.0525	.0555	.0585	.0615	.0645	.0675	.0705	.0735	.0765	.0795	
.012				1	2		1						4
.014	0.5	1	3.5	2.5	4	3	2						16.5
.016	1	3	15	15.5	8	10	5	6	3	2	1	1	70.5
.018	1	1	1	10	13	11.5	22.5	17.5	7		3		87.5
.020			3	1	3	5.5	4	2	1	3.5			23
.022			1		1	3	1	3		1	1		11
.024							1.5	1.5	1	0.5			4.5
.026													0
.028								1					1
	2.5	5	23.5	30	31	33	37	31	12	7	5	1	218

$$M_3 = .0615, \quad M_4 = .01752.$$

$$\sigma_3 = .0066, \quad \sigma_4 = .00232.$$

$$r_{34} = 0.333.$$



No. 3.

No. 5.													
	.0465	.0495	.0525	.0555	.0585	.0615	.0645	.0675	.0705	.0735	.0765	.0795	
.005		1	2	4		3	3	1		1		15	
.015			3	3	2	3	7.5	2.5		1	1	23	
.025	0.5	2	1.5	2	5.5	2.5	4.5	8.5	2	1	1	31	
.035			4	3	1.5	5.5	2	3	1		1	22	
.045		1.5	4.5	1.5	7	7	7	2	4			34.5	
.055	1	0.5	3	4	1	4	6	5	2	1		27.5	
.065	1		1.5	5.5	3	3	3	5	1	2	1	26	
.075			4	5	5.5	4	1	1	1			21.5	
.085				2	3.5	1	2	3	1		1	13.5	
.095							1			1		2	
.105					2							2	
	2.5	5	23.5	30	31	33	37	31	12	7	5	1	218

$$M_3 = .0615. \quad M_5 = .04532.$$

$$\sigma_3 = .0066. \quad \sigma_5 = .024.$$

$$r_{35} = -0.0082.$$



No. 4.

No. 5.

	.012	.014	.016	.018	.020	.022	.024	.026	.028	
.005			8	6	1					15
.015	1	2	4	7.5	5.5	2	1			23
.025		3	9	13	3	1	2			31
.035		1	5.5	11.5	3				1	22
.045	1	2	12.5	15	3	1				34.5
.055		3	9	10.5	2	2.5	0.5			27.5
.065		1.5	7.5	11	2	4				26
.075	1	2	10.5	5	3					21.5
.085		1	4.5	7			1			13.5
.095				1	0.5	0.5				2
.105	1	1								2
	4	16.5	70.5	87.5	23	11	4.5	0	1	218

$$M_4 = .01752. \quad M_5 = .04532.$$

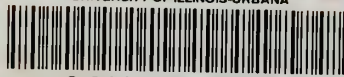
$$\sigma_4 = .00232. \quad \sigma_5 = .024.$$

$$r_{45} = -0.0907.$$





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